

# AN INTEGRAL EQUATION APPROACH FOR ANALYSIS OF CONTROL CHARTS

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A thesis submitted for the degree of  
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in Mathematical Sciences

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I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Yupaporn Areepong  
Signature of Candidate



# Dedication

*To Dad & Mom*

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# Table of Contents

Table of Contents	vi
List of Figures	viii
List of Tables	ix
Abstract	x
<b>1 Introduction</b>	<b>1</b>
1.1 Background to Problem . . . . .	2
1.2 Research Objectives and Contributions . . . . .	3
1.3 Scope of Study . . . . .	4
1.4 Outline of Thesis . . . . .	4
1.5 List of Publications and Presentations . . . . .	5
<b>2 Statistical Process Control Charts</b>	<b>6</b>
2.1 Chart Characteristics . . . . .	6
2.2 Shewhart Procedure . . . . .	7
2.3 EWMA Procedure . . . . .	9
2.4 CUSUM Procedure . . . . .	13
2.5 Shirayev-Roberts Procedure . . . . .	15
2.6 Discussion . . . . .	15
<b>3 Methods for Evaluating the Performance of EWMA</b>	<b>17</b>
3.1 Markov Chain Approach . . . . .	17
3.2 Integral Equations . . . . .	23
3.3 Monte Carlo Simulations . . . . .	27
3.4 Discussion . . . . .	29
<b>4 EWMA Control Charts for Change in Exponential Distribution</b>	<b>30</b>
4.1 The Exponential Family of Distributions . . . . .	30
4.2 Explicit Formulas for One-sided Exponential EWMA . . . . .	32
4.3 Comparison of Numerical Results for $E_x(\theta^{\tau_b})$ . . . . .	40
4.4 Comparison of Numerical Results for ARL and AD with Other Methods . .	43

4.5 Comparison of EWMA, CUSUM and Shirayev-Roberts Charts: Exponential case . . . . . 44

4.6 Choices of Optimal Parameters for Exponential EWMA Charts . . . . . 46

4.7 Explicit Formulas for One-sided Pareto EWMA . . . . . 49

**5 Numerical Solutions for Integral Equations . . . . . 54**

5.1 The Nyström Method . . . . . 55

5.2 Integral Equations for EWMA Chart . . . . . 68

5.3 Integral Equations for CUSUM Chart . . . . . 77

**6 Conclusion and Recommendations for Further Research . . . . . 81**

6.1 Overall Conclusion . . . . . 81

6.2 Recommendations for Further Research . . . . . 82

**A Codes for Simulation and Calculation by Mathematica® . . . . . 83**

A.1 Exponential case . . . . . 83

A.2 Pareto case . . . . . 85

A.3 Gamma case . . . . . 88

A.4 Weibull case . . . . . 90

**B Mathematica® Codes for Algorithms of Obtaining Optimal Designs of EWMA . . . . . 94**

B.1 Procedure for obtaining optimal parameter values . . . . . 94

B.2 Calculation codes for obtaining numerical method for Integral Equations . . 96

B.3 Calculation codes for obtaining numerical results of MCA . . . . . 100

**Bibliography . . . . . 102**



# List of Figures

2.1	Shewhart chart . . . . .	8
2.2	EWMA charts . . . . .	12
3.1	In-control region divided into N subintervals of the Markov Chain . . . . .	21
4.1	Curves of AD for EWMA, CUSUM and Shirayayev-Roberts charts: exponential case . . . . .	46
4.2	Curves of AD for optimal exponential EWMA designs . . . . .	47
4.3	Curves of AD for EWMA, CUSUM and Shirayayev-Roberts charts: Pareto case . . . . .	52
4.4	Curves of AD for optimal Pareto EWMA designs . . . . .	52
5.1	Curves of AD for EWMA, CUSUM and Shirayayev-Roberts charts: gamma case . . . . .	70
5.2	Curves of AD for optimal gamma EWMA designs . . . . .	72
5.3	Curves of AD for EWMA, CUSUM and Shirayayev-Roberts charts: Weibull case . . . . .	74
5.4	Curves of AD for optimal Weibull EWMA designs . . . . .	76



# List of Tables

4.1	Comparison of the numerical results for $E_x(\theta^{\tau_b})$ . . . . .	43
4.2	Comparison of ARL and AD values with other methods . . . . .	44
4.3	The numerical results for ARL and AD obtained from formula (4.33) and (4.39) and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: exponential case . . . . .	45
4.4	Optimal design parameters and AD for one-sided exponential EWMA . . .	48
4.5	The numerical results for ARL and AD obtained from formulas (4.59) and (4.60) and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: Pareto case . . . . .	51
4.6	Optimal design parameters and AD for one-sided Pareto EWMA . . . . .	53
5.1	Comparison of approximations for ARL and AD from Integral Equations and Monte Carlo Simulation . . . . .	67
5.2	Comparison of approximations ARL and AD obtained from Integral Equations with MCA and Monte Carlo Simulation: gamma case . . . . .	69
5.3	The numerical results for ARL and AD obtained from Integral Equation and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: gamma case . . . . .	70
5.4	Optimal design parameters and AD for one-sided gamma EWMA . . . . .	71
5.5	Comparison of approximate ARL and AD values obtained from Integral Equations with MCA and Monte Carlo Simulation: Weibull case . . . . .	73
5.6	The numerical results for ARL and AD obtained from Integral Equation and simulation results of EWMA, CUSUM and Shiryayev-Roberts charts: Weibull case . . . . .	74
5.7	Optimal design parameters and AD for one-sided Weibull EWMA . . . . .	75
5.8	Comparison of approximation ARL and AD values obtained from Integral Equation for two-sided EWMA with values obtained from Monte Carlo Simulation: exponential case . . . . .	77
5.9	Comparison of approximation ARL and AD values obtained from Integral Equation with values from Monte Carlo Simulation: exponential CUSUM case . . . . .	79
5.10	Comparison of approximation ARL and AD values obtained from Integral Equation with values from Monte Carlo Simulation: gamma CUSUM case .	80

# Abstract

This thesis is concerned with the use of Statistical Process Control (SPC) charts for detection of change-point in distributions in quality control and surveillance problems. We derive explicit analytical formulas and develop numerical algorithms for evaluating important characteristics of “Exponentially Weighted Moving Average” (EWMA) control charts for a range of distributions.

The most popular characteristics of a control chart are Average Run Length (ARL) - the mean of observations/times that are taken before a system is signalled to be out-of-control when it is actually still in-control, and Average Delay (AD) time - the mean of delay of true alarm times before a system that is actually out-of-control is signalled to be out-of-control. An important property required of ARL is that it should be sufficiently large when the process is in-control to reduce a number of false alarms. On the other side, if the process is actually out-of-control then its AD should be as small as possible. Traditional methods that are used for evaluating chart characteristics include Markov Chain Approach (MCA), Integral Equation (IE) and Monte Carlo simulation (MC) methods. Some crucial features of the methods are as follows: the MCA requires many matrix inversions and there is no theoretical proof of convergence of the method; the IE is most advanced method and it was used before only for Gaussian distribution; the MC is very time consuming.

In this thesis, we find explicit formulas for ARL and AD of EWMA in the case when observations are exponentially distributed. These explicit formulas can be applied to some other distributions, e.g. the Pareto distribution. The numerical results obtained from our explicit formulas are compared with results obtained from the Monte Carlo simulation (MC) and Markov Chain Approach (MCA). We also compare the performance of the EWMA procedure with charts obtained with the CUSUM and Shirayev-Roberts procedures. The technique that we use to derive the formulas for an exponential distribution cannot be used to derive formulas for gamma and Weibull distributions. However, we have developed a different method for evaluating the ARL and AD for the case of gamma and

Weibull distributions. This method is based on a numerical solution of Integral Equations based on Gauss-Legendre integration rules to approximate the integrals. Numerical results for these distributions are compared with results from other approaches.